On Charge Conservation and The Equivalence Principle in the Noncommutative Spacetime

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Abstract

We investigate one of the consequences of the twisted Poincaré symmetry. We derive the charge conservation law and show that the equivalence principle is satisfied in the canonical noncommutative spacetime. We applied the twisted Poincaré symmetry to the Weinberg's analysis [11]. To this end, we generalize our earlier construction of the twisted S matrix [10], which apply the noncommutativity to the fourier modes, to the massless fields of integer spins. The transformation formula for the twisted S matrix for the massless fields of integer spin has been obtained. For massless fields of spin 1, we obtain the conservation of charge, and the universality of coupling constant for massless fields of spin 2, which can be interpreted as the equality of gravitational mass and inertial mass, i.e., the equivalence principle.

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1 Introduction

In effort to construct an effective theory of quantum gravity at the Planck scale, noncommutativity of the spacetime has been considered. The canonical noncommutative spacetime has the commutation relations between the coordinates [1],

$$[x^\mu, x^\nu] \ = \ i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}(\mu,\nu = 0 \text{ to } 3)$ is a constant antisymmetric matrix.

Field theories in the canonical noncommutative spacetime can be replaced by field theories in the commutative spacetime with the Moyal product (The Weyl-Moyal correspondence [2]). One of the significant problems of those theories is that they violate the Lorentz symmetry. One finds that the symmetry group is $SO(1,1)\times SO(2)$ instead of the Lorentz group, $SO(1,3)$. Since there is no spinor or vector representations in that symmetry group, most of the earlier studies performed by using the spinor, vector representations of the Lorentz group can not be justfied. Moreover, the factors $1(-1)$ are multiplied for a boson(fermion)loop without knowing the spin-statistics relation.

To get around this, Chaichian et.al.* have deformed the Poincaré symmetry as well as its module space to which the symmetry acts [3]. The twisted symmetry group has the same representations as the original Poincaré group and at the same time they successfully retain the physical information of the canonical noncommutativity. The main idea was that one can change a classical symmetry group to a quantum group, $ISO_\theta(1,3)$ in this case, and twist-deform the module algebra consistently to reproduce the noncommutativity. In their approach, the noncommutative parameter $\theta^{\mu\nu}$ transform as an invariant tensor. This reminds us the situation that Einstein had to change the symmetry group of the spacetime and its module space(to the Minkowski spacetime) when the speed of light is required to be constant for any observer in an inertial frame. Similarly, Chaichian et.al. have required the change of the Hopf algebra with its module algebra so that any observer in an inertial frame feel the noncommutativity in the same way. For the κ-deformed noncommutativity, Majid and Ruegg found the κ-deformed spacetime [6] as a module space of the κ-deformed symmetry after Lukierski et.al. discovered the symmetry [7]. The real benefit of the twist is in the use of the same irreducible representations of original theories unlike general deformed theories, as in the case of the κ-deformed theory.

Recently, groups of physicists have constructed the quantum field theory in the noncommutative spacetime by twisting the quantum space as a module space [8],[9],[10]. Espcially, Bu et.al. have proposed a twisted S-matrix as well as a twisted Fock space for consistency [10]. There we have obtained the twisted algebra of the creation and annihilation operators and the spin-statistics relation by applying the twisted Poincaré symmetry on the quantum space consistently. The analysis of this paper is mainly based on this work.

These works justify the use of the irreducible representations of the Poincaré group and the sign factors being used in the earlier studies. Are these works merely a change of viewpoint? Mathematically they look equivalent and seem to have equal amounts of information. But when the physics

* Oeckl [4], Wess [5] have proposed the same deformed Poincaré algebra.

is concerned the action of symmetry becomes more subtle than it seems because it confines possible configurations of physical systems. In this article, we present an example showing the role of the twisted symmetry for solving physics problems, especially in the canonical noncommutative spacetime. As an example, we derive the conservation law of charge and show the fact that the equivalence principle is satisfied even in the noncommutative spacetime. In this derivation we consider spin 1 and 2 massless fields for the photon and the graviton, respectively. For this purpose, we extend our earlier study on the scalar field theory to more general field theories and investigate a noncommutative version of Weinberg's analysis [11, 12, 13].

Actually, there are many studies for the relation between noncommutativity and gauge theory [14], [15], [16], [17] or between the noncommutativity and the gravity [18],[19],[20] and between them [21]. And there were many argument whether the equivalence principle is satisfied at the quantum level. Some people argued that the equivalence principle is violated in quantum regime [22],[23], while there are studies which show non-violation of the equivalence principle [24]. Whether the principle of equivalence is violated or not is an important issue for quantum gravity because the principle is the core of the general relativity.

The paper is organized as follows. We extend our previous construction of the $S_\star$ matrix to the massless fields of integer spin after giving a brief review on the construction and the properties of the $S_\star$ matrix. We give an exact transformation formula for the $S_\star$ matrix elements in section 2. We give the consequence of requiring the twist invariance to the $S_\star$ matrix elements for the scattering process. These results lead to the charge conservation law for the spin 1 field theory and the universality of the coupling constant for spin 2 field in noncommutative spacetime in section 3. Finally, we discuss the implication of the twisted symmetry and its applicability to other issues in section 4. We give some related calculations of the polarization vector, the noncommutative definition of the invariant M function, and a twisted transformation formula for the $S_\star$ matrix in the Appendices.

2 Properties of the general $S_\star$ matrix

2.1 A short introduction of useful properties of the twist-deformation

An algebra with a product $\cdot$ and a coalgebra with a coproduct Δ constitute a Hopf algebra if it has an invertible element S called antipode and with some compatibility relations. For a Lie algebra **g**, there is a unique universal enveloping algebra $\mathcal{U}(\mathbf{g})$ which preserves the Lie algebra properties in terms of unital associative algebra. The Hopf algebra of a Lie algebra **g** is denoted as $\mathcal{H} \equiv \{\mathcal{U}(\mathbf{g}), \cdot, \Delta, \epsilon, S\}$, where $\mathcal{U}(\mathbf{g})$ is an universal enveloping algebra of the corresponding algebra **g** and we denotes the counit as ϵ. The Sweedler notation is being widely used for a shorthand notation of the coproduct, $\Delta Y = \sum Y_{(1)} \otimes Y_{(2)}$ [25].

The action of a Hopf algebra $\mathcal{H}$ to a module algebra $\mathcal{A}$ is defined as

$$Y \triangleright (a \cdot b) = \sum (Y_{(1)} \triangleright a) \cdot (Y_{(2)} \triangleright b), \tag{2}$$

where $a, b \in \mathcal{A}$, the symbol $\cdot$ is a multiplication in the module algebra $\mathcal{A}$, and the symbol $\triangleright$ denotes an action of the Lie generators $Y \in U(\mathbf{g})$ on the module algebra $\mathcal{A}$. The product $\cdot$ in $\mathcal{H}$ and the

multiplication $\cdot$ in $\mathcal{A}$ should be distinguished.

If there is an invertible 'twist element', $\mathcal{F} = \sum \mathcal{F}_{(1)} \otimes \mathcal{F}_{(2)} \in \mathcal{H} \otimes \mathcal{H}$, which satisfies

$$\begin{aligned}(\mathcal{F} \otimes 1) \cdot (\Delta \otimes \mathrm{id})\mathcal{F} &= (1 \otimes \mathcal{F}) \cdot (\mathrm{id} \otimes \Delta)\mathcal{F}, \qquad (3)\\ (\epsilon \otimes \mathrm{id})\mathcal{F} = \quad 1 &= (\mathrm{id} \otimes \epsilon)\mathcal{F}, \qquad (4)\end{aligned}$$

one can obtain a new Hopf algebra $\mathcal{H}_{\mathcal{F}} \equiv \{\mathcal{U}_{\mathcal{F}}(\mathbf{g}), \cdot, \Delta_{\mathcal{F}}, \epsilon_{\mathcal{F}}, S_{\mathcal{F}}\}$ from the original one. The relations between them are

$$\begin{aligned}\Delta_{\mathcal{F}} Y = \mathcal{F} \cdot \Delta Y \cdot \mathcal{F}^{-1} \quad &, \quad \epsilon_{\mathcal{F}}(Y) = \epsilon(Y),\\ S_{\mathcal{F}}(Y) = u \cdot S(Y) \cdot u^{-1} \quad &, \quad u = \sum \mathcal{F}_{(1)} \cdot S(\mathcal{F}_{(2)}), \qquad (5)\end{aligned}$$

with the same product in the algebra sector. The 'covariant' multiplication of the module algebra $\mathcal{A}_{\mathcal{F}}$ for the twisted Hopf algebra $\mathcal{H}_{\mathcal{F}}$ which maintain the form of Eq.(2) is given as

$$(a \star b) = \cdot[\mathcal{F}^{-1}\, (a \otimes b)]. \qquad (6)$$

From the above relations, one can derive an important property of the twist such that it does not change the representations of the algebra:

$$\begin{aligned}D_{\mathcal{F}}(Y)(a \star b) &= \star\, [\Delta_{\mathcal{F}} Y\, (a \otimes b)]\\ &= \cdot\, [\mathcal{F}^{-1} \cdot \mathcal{F} \Delta_0 Y \mathcal{F}^{-1} (a \otimes b)]\\ &= \cdot\, [\Delta_0 Y\, \mathcal{F}^{-1} (a \otimes b)]\\ &= D_0(Y)(a \star b), \qquad (7)\end{aligned}$$

where representations of the coproduct and the twist element is implied, i.e.,

$$D[\Delta Y] = \sum D(Y_{(1)}) \otimes D(Y_{(2)}), \quad D[\mathcal{F}] = \sum D(\mathcal{F}_{(1)}) \otimes D(\mathcal{F}_{(2)}). \qquad (8)$$

The above considerations lead us to the golden rule: *The irreducible representations are not changed by a twist and one can regard the covariant action of a twisted Hopf algebra on a twisted module algebra as the action of the original algebra on the twisted module algebra.*

2.2 The $S_\star$ matrix and its twist invariance

Recently, a quantum field theory has been constructed in such a way to preserve the twisted Poincaré symmetry [10]. There we confined the construction for the space-space noncommutativity. It is hard to know whether one can construct a consistent twist Poincaré invariant field theory satisfying the causality in the case of space-time noncommutativity. They have tried to apply the twisted symmetry to quantum spaces consistently, especially to the algebra of the creation and annihilation operators ($a_p^\dagger$ and a_p). As a result, they obtained the twisted algebra of quantum operators. If we use a shorthand notation, $p \wedge q = p_\mu \theta^{\mu\nu} q_\nu$, the twisted algebra of $a_p^\dagger$ and a_p can be denoted as:

$$c_p \star\; c_q = e^{-\frac{i}{2}\tilde{p}\wedge\tilde{q}}\; c_p \cdot\; c_q, \qquad (9)$$

where c_p can be a_p or $a_p^\dagger$, $\widetilde{p} \equiv -p(p)$ for $c_p = a_p(a_p^\dagger)$ and $\cdot$ denotes the ordinary multiplication of operators in the commutative theories.

This twisted algebra naturally leads to the twisted form of Fock space, S-matrix and quantities related to the creation and annihilation operators. Thus, we obtain the twisted basis of Fock space and $S_\star$-matrix:

$$|q_1, \cdots, q_n\rangle \to |q_1, \cdots, q_n\rangle_\star = \mathcal{E}(q_1, \cdots, q_n)|q_1, \cdots, q_n\rangle, \tag{10}$$

where $\mathcal{E}(q_1, \cdots, q_n) = \exp\left(-\frac{i}{2}\sum_{i<j}^n q_i \wedge q_j\right)$ is a phase factor which has the interesting properties [10], and

$$S \to S_\star = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \int d^4x_1 \cdots d^4x_k \, T\left\{\mathcal{H}_I^\star(x_1) \star \cdots \star \mathcal{H}_I^\star(x_k)\right\}, \tag{11}$$

where T denotes the time ordering and $\mathcal{H}_I^\star(x)$ is an interaction Hamiltonian density in the Dyson formalism.

The explicit form of the $S_\star$ matrix elements for the scalar ϕ^n theoryin the momentum space is:

$${}_\star\langle\beta|S_\star|\alpha\rangle_\star = \mathcal{E}(-\beta, \alpha)\sum_{k=0}^{\infty}(-ig)^k \int_{Q_1} \cdots \int_{Q_k} \sum_{c_{Q_1}\cdots c_{Q_k}} \mathcal{E}(\widetilde{Q}_1)\cdots\mathcal{E}(\widetilde{Q}_k)\langle\beta|S^k(\widetilde{Q}_1 \cdots, \widetilde{Q}_k)|\alpha\rangle, \tag{12}$$

where $\widetilde{Q}$ is the shorthand notation for $(\widetilde{q}_1, \ldots, \widetilde{q}_n)$ [10].

In the above, $\langle\beta|S^k(\widetilde{Q}_1 \cdots, \widetilde{Q}_k)|\alpha\rangle$ is a g^k order term of the S-matrix element of the commutative theory where g is the coupling constant of the theory. From the momentum conservation, i.e., delta functions in the $\langle\beta|S^k(\widetilde{Q}_1 \cdots, \widetilde{Q}_k)|\alpha\rangle$, one can show that the $S_\star$ matrix element, ${}_\star\langle\beta|S_\star|\alpha\rangle_\star$, can be represented by Feynman diagrams with extra phase factors $\mathcal{E}(\widetilde{Q})$ for each vertex. The phase factors drastically change the predictions of the theory. This result agrees with Filk's result[26], but we have overall factors $\mathcal{E}(-\beta, \alpha)$ corresponding to external lines in the Feynman diagram which originated from the twisted Fock space. From the above considerations, the new modified Feynman diagrams can be obtained from the untwisted ones by changing the phase factors from 1 to $\mathcal{E}(\widetilde{Q}_i)$ at each vertex.

The twist invariance of this prescription of the $S_\star$ matrix is not manifest because non-locality of the interactions may violate the twist invariance of the $S_\star$ matrix, in general, i.e.,

$$[\mathcal{H}_I^\star(x), \mathcal{H}_I^\star(y)]_\star \neq 0 \ \text{ for spacelike } (x-y). \tag{13}$$

However, we see from the form of $S_\star$ matrix in Eq.(12), that the proposed $S_\star$ matrix is clearly twist invariant since it is constructed from phase factors which are twist invariant, and the Feynman propagators. Twisted product of fields operators satisfy

$$\langle 0|\psi(x) \star \psi(y)|0\rangle = \langle 0|\psi(y) \star \psi(x)|0\rangle, \ \text{for spacelike } (x-y). \tag{14}$$

Hence we see that the Feynman propagator, same as the twist Feynman propagator $\langle 0|T[\psi(x) \star \psi(y)]|0\rangle$, is twist invariant. From this, the invariance of the $S_\star$ matrix elements follows immediately.

2.3 Generalization to arbitrary fields

We need to get the $S_\star$ matrix for massless field theories of integer spin for the analysis of this paper. In the previous work [10], we have constructed the $S_\star$ matrix for scalar field theory and we have expected that the same formulation could be possible for general field theories. In this section, we generalize our argument used in that paper to obtain the form of the $S_\star$ matrix elements for massless field theories with spin 1 and 2. In the analysis of this paper, we use the $(s/2, s/2)$ representation for massless integer spin fields. The reason we use it and the considerations for the other representations are given in section 4.

We have used the perturbation theory in our formulation of the $S_\star$ matrix in Eq.(11). Another assumption was the particle interpretation. That is, the field operators are represented as linear combinations of the creation and annihilation operators and the fields of spin s transform(twist) as[†] :

$$[D^s_\theta(\Lambda^{-1})]^A_{\ B}\ \hat{\Psi}_\theta{}^B(\Lambda x + a) = U_\theta(\Lambda, a)\ \hat{\Psi}_\theta{}^A(x)\ U_\theta^{-1}(\Lambda, a), \tag{15}$$

where D^s denotes the irreducible representation for spin s. Since translations act homogenously on the fields, twisted tensor fields can be written as

$$\hat{\Psi}_\theta{}^A(x) = \int_p \sum_\sigma [a_\sigma(p)\ \epsilon_\theta{}^A(p,\sigma)\ e^{ip\cdot x} + b^\dagger_\sigma(p)\ \chi_\theta{}^A(p,\sigma)\ e^{-ip\cdot x}], \tag{16}$$

where A denotes the tensor index, $A \equiv (a_1 \cdots a_s)$ for massless spin s fields and σ for the helicity indices. Thus, the transformation relation of the fields, Eq.(15), can be reduced to

$$[D^s_\theta(\Lambda^{-1})]^A_{\ B}\ \hat{\Psi}_\theta{}^B(\Lambda x) = U_\theta(\Lambda)\ \hat{\Psi}_\theta{}^A(x)\ U_\theta^{-1}(\Lambda). \tag{17}$$

In order that the field operators transform correctly, the 'polarization' tensor $\epsilon_\theta{}^A(p,\sigma)$ are to be transformed as

$$\begin{aligned}
\epsilon_\theta{}^A(p,\sigma) &= \frac{1}{(2\pi)^3\sqrt{2\omega_p}}\,[D^s_\theta(\mathcal{L}(p))]^A_{\ B}\ \epsilon_\theta{}^B(k,\sigma), \\
\chi_\theta{}^A(p,\sigma) &= \frac{1}{(2\pi)^3\sqrt{2\omega_p}}\,[D^s_\theta(\mathcal{L}(p))]^A_{\ B}\ \chi_\theta{}^B(k,\sigma')\ C^{-1}_{\sigma'\sigma},
\end{aligned} \tag{18}$$

where $\mathcal{L}(p)$ is the Lorentz transformation[‡] which reflects the little group and C is a matrix with the properties [12],

$$C^*\ C = (-1)^{2s}, \quad C^\dagger\ C = 1. \tag{19}$$

[†] Recently, Joung and Mourad [27] have argued that a covariant field linear in creation and annihilation operators does not exist. But the definition of creation and annihilation operators in that paper is different from ours. We try to follow the view that we use the same fundamental quantities as of those in the untwisted theories changing the algebras only. The field operators in linear combinations of the creation and annihilation operators can be justified in this basis, because the free fields are the same as the commutative case.

[‡] $\mathcal{L}(p)$ is a Lorentz transformation that takes the standard momentum k^μ to $p^\mu \equiv (|\mathbf{p}|, \mathbf{p})$ for massless fields. For the massless case $\mathcal{L}(p)$ satisfies $\mathcal{L}(p) = R(\hat{p})B(|\mathbf{p}|)$ where $R(\hat{p})$ is a rotation which takes the direction of the standard momentum $\mathbf{k}$ to the direction of $\mathbf{p}$ and $B(|\mathbf{p}|)$ is the boost along the $\mathbf{p}$ direction [11].

Since the most important properties of the twist is that the representations are the same as in the original group, the Poincaré group in this case, the above $D^s_\theta(\Lambda^{-1})$ and $D^s_\theta(\mathcal{L})$ can be replaced by $D^s(\Lambda^{-1})$ and $D^s(\mathcal{L})$, respectively. Thus the θ-dependance remains only in the polarization tensors. Furthermore, one can see that there's no θ-dependance in ${\epsilon_\theta}^A(\sigma)$ and ${\chi_\theta}^A(\sigma')$. We can show this by considering the case of ${\epsilon_\theta}^A(\sigma)$. From the group property of the twisted Lorentz transformations, the related transformation to the polarization tensor ${\epsilon_\theta}^A(\sigma)$ can be written as:

$$[D^s_\theta(\Lambda^{-1})]^A_{\ B}\,[D^s_\theta(\mathcal{L})]^C_{\ D}\,{\epsilon_\theta}^D(\sigma) = [D^s_\theta(\Lambda^{-1}\cdot\mathcal{L})]^A_{\ B}\,{\epsilon_\theta}^B(\sigma). \tag{20}$$

In order for this transformation to be a twist transformation, it is to be satisfied

$$[D^s_\theta(\Lambda^{-1}\cdot\mathcal{L})]^A_{\ B}\,{\epsilon_\theta}^B(\sigma) = [D^s(\Lambda^{-1}\cdot\mathcal{L})]^A_{\ B}\,{\epsilon_\theta}^B(\sigma), \tag{21}$$

i.e., twisted transformation has the same representation as the untwisted one. From the primary relation between the twist and the module algebra, one can see that the form of ${\epsilon_\theta}^A(\sigma)$, twisted version of the commutative polarization tensor, should be

$${\epsilon_\theta}^{a_1\cdots a_n}(\sigma) = \quad\cdot\,[\mathcal{F}_n^{-1}\triangleright\epsilon^{a_1}(\sigma)\otimes\cdots\otimes\epsilon^{a_n}(\sigma)], \tag{22}$$

where $\mathcal{F}_n^{-1}$ can be obtain from[§] the $\mathcal{F}_\theta$, and $\mathcal{F}_\theta$ is the very twist element corresponding to the canonical noncommutativity:

$$\mathcal{F}_\theta = \exp\left(\frac{i}{2}\theta^{\alpha\beta}P_\alpha\otimes P_\beta\right). \tag{23}$$

Since $P_\alpha\triangleright\epsilon^a(\sigma)=0$, we obtain

$${\epsilon_\theta}^{a_1a_2\cdots a_s}(\sigma) \equiv \epsilon^{\ a_1a_2\cdots a_s}(\sigma) = \epsilon^{(a_1}_\sigma\epsilon^{a_2}_\sigma\cdots\epsilon^{a_s)}_\sigma, \tag{24}$$

where $\epsilon^{(a_1a_2\cdots a_s)}(\sigma)$ is the polarization tensor in the corresponding commutative field theory. This relation leads to the relation

$$\hat{\Psi}_\theta{}^A(x) \equiv \hat{\Psi}^A(x). \tag{25}$$

This relation is expected because the twist does not change the representations and we write the field operators by using an irreducible representations of the symmetry group. Explicit calculations for massless $(s/2, s/2)$ tensor representation which shows the non-dependance on the θ is given in the Appendix A.

Therefore, one can safely use the same representations for the field operators in the twisted theory as those of the untwisted ones. What really be twisted are the multiplications of the creation and annihilation operators only. Since the multiplication of the creation and annihilation operators between different species of particles act like composite mappings on the Hilbert space, one can construct a

[§] The associativity of the twisted product $*$ guaranty that it can be obtained by successive applications of Eq.(6).

module algebra from them. That is, when c_p's and d_p's are the creation and annihilation operators corresponding to different species, their twisted multiplications are

$$c_p \star\ d_q = e^{-\frac{i}{2}\widetilde{p}\wedge\widetilde{q}}\ c_p \cdot\ d_q, \tag{26}$$

where $c_p(d_p)$ can be $a_p(b_p)$ or $a_p^\dagger(b_p^\dagger)$, and $\widetilde{p}$ and $\widetilde{q}$ are defined as in section 2.2.

Consequently, the scheme for twisting S-matrix used in [10] applies for general field theories. Twist invariance of the $S_\star$ matrix for general field theories follows immediately. As in the scalar field theory, the amplitudes can be obtained by multiplying the phase factor $\mathcal{E}(q_1, \cdots, q_n)$ for each vertex in the Feynman diagram of the untwisted theories.

2.4 Exact transformation formula for the $S_\star$ matrix element for massless fields

Transformation formula for the $S_\star$ matrix element corresponding to the process in which a massless particle is emitted with momentum **q** and helicity s can be inferred as (Appendix C)

$$S_\star^{\pm s}(\mathbf{q}, p) = \sqrt{\frac{|\Lambda\mathbf{q}|}{|\mathbf{q}|}}\exp\left[\pm is\Theta(\mathbf{q}, \Lambda)\right] S_\star^{\pm s}(\Lambda\mathbf{q}, \Lambda p). \tag{27}$$

The $S_\star$ matrix can be written as the scalar product of a polarization tensor and the $M_\star$ function(Appendix C):

$$S_\star^{\pm s}(\mathbf{q}, p) = \frac{1}{\sqrt{2|\mathbf{q}|}}\epsilon_\pm^{\mu_1 *}(\hat{q}) \cdots \epsilon_\pm^{\mu_s *}(\hat{q})(M_\star^\pm)_{\mu_1 \cdots \mu_s}(\mathbf{q}, p), \tag{28}$$

where the $M_\star$ function twist-transform covariantly as

$$M_\star^{\pm\mu_1\cdots\mu_s}(\mathbf{q}, p) = \Lambda_{\nu_1}^{\ \mu_1} \cdots \Lambda_{\nu_s}^{\ \mu_j} M_\star^{\pm\nu_1\cdots\nu_s}(\Lambda\mathbf{q}, \Lambda p). \tag{29}$$

The form of the $S_\star$ matrix element in Eq.(28) appears to break the twisted Poincaré symmetry because the polarization vectors do not satisfy the Lorentz covariance, rather they satisfy

$$(\Lambda_\nu^{\ \mu} - q^\mu \Lambda_\nu^{\ 0}/|\mathbf{q}|)\epsilon_\pm^\nu(\Lambda q) = \exp\left\{\pm i\Theta(\mathbf{q}, \Lambda)\right\} \epsilon_\pm^\mu(\hat{q}). \tag{30}$$

Hence requiring the twist invariance of the $S_\star$ matrix would lead to a constraint relation between the momentum, the polarization vectors and the $M_\star$ function. From Eq.(27) and Eq.(30), the $S_\star$ matrix element in Eq.(28) can be written as

$$\begin{aligned} S_\star^{\pm s}(\mathbf{q}, p) \quad &= \quad \frac{1}{\sqrt{2\mathbf{q}}}\exp\left[\pm is\Theta(\mathbf{q}, \Lambda)\right]\left[\epsilon_\pm^{\mu_1}(\Lambda q) - (\Lambda q)^{\mu_1}\Lambda_\nu^{\ 0}\epsilon_\pm^\nu(\Lambda q)/|\mathbf{q}|\right]^* \\ &\qquad \cdots \left[\epsilon_\pm^{\mu_s}(\Lambda q) - (\Lambda q)^{\mu_s}\Lambda_\nu^{\ 0}\epsilon_\pm^\nu(\Lambda q)/|\mathbf{q}|\right]^*(M_\star)_{\pm\mu_1\cdots\mu_s}(\Lambda\mathbf{q}, \Lambda p). \end{aligned} \tag{31}$$

Requiring the twist invariance of the $S_\star$ matrix element results in:

$$q^{\mu_1}\epsilon_\pm^{\mu_2 *}(\hat{q}) \cdots \epsilon_\pm^{\mu_s *}(\hat{q}) M^\pm_{\star\ \mu_1\cdots\mu_s} = 0. \tag{32}$$

This leads us to the desired identities:

$$q_\rho M_\star^{\pm\rho\mu_2\cdots\mu_s}(\mathbf{q}, p) = 0. \tag{33}$$

3 Charge conservation and the equivalence principle

Since the analysis of the conservation law is just a noncommutative generalization of Weinberg's work [11], the derivation of this section will be fairly straightforward. As we saw in section 2, the differences between the noncommutative field theory and the commutative one are the phase factors at each vertex of the Feynman diagrams.

3.1 Dynamical definition of charge and gravitational mass

We define the charge and the gravitational mass dynamically as the coupling of the vertex amplitudes for the soft(very low energy) photon and graviton, respectively. Consider the vertex amplitude for the process that a soft[¶] massless particle of momentum q and spin s is emitted by a particle of momentum p and spin J. Since the only tensor which can be used to form the invariant M function is known to be $p^{\mu_1} \cdots p^{\mu_s}$, the noncommutative M function, $M_\star$ function, is given by the commutative M function with phase factor $\mathcal{E}(\widetilde{q}, \widetilde{p}, \widetilde{p}')$ multiplied at each vertex. That is, the vertex amplitude can be written as

$$\Gamma_\star \propto \frac{\mathcal{E}(\widetilde{q}, \widetilde{p}, \widetilde{p}')}{2E(\mathbf{p})\sqrt{2|\mathbf{q}|}}\, p_{\mu_1} \cdots p_{\mu_s}\, \epsilon_\pm^{\mu_1 *}(\hat{q}) \cdots \epsilon_\pm^{\mu_s *}(\hat{q}). \tag{34}$$

When the emitting particle has spin J we have to multiply to the vertex amplitude $\delta_{\sigma\sigma'}$ [11]. Thus, the explicit form of the vertex amplitudes can be written as:

$$\frac{2i(2\pi)^4 \cdot \mathbf{e_s} \cdot \delta_{\sigma\sigma'}[p_\mu\, \epsilon_\pm^{\mu *}(\hat{q})]^s}{(2\pi)^{9/2}[2E(\mathbf{p})]\sqrt{2|\mathbf{q}|}} \cdot \mathcal{E}(\widetilde{q}, \widetilde{p}, \widetilde{p}'), \tag{35}$$

where $\mathbf{e_s}$ is a coupling constant for emitting a soft massless particle of spin s (e.g., photon and graviton).

These coupling constants for emitting a soft particle can be interpreted as: $\mathbf{e_1} \equiv e$ as the electric charge, and $\mathbf{e_2} \equiv \sqrt{8\pi G}g$, with g the ratio of the gravitational mass and the inertial mass. Let us consider a near forward scattering of the two particles A and B with the coupling $\mathbf{e}_A^s$ and $\mathbf{e}_B^s$, respectively. From the properties of the phase factors [10], the phase factor for this scattering, $\mathcal{E}(\widetilde{q}, \widetilde{p}_A, \widetilde{p}_{A'}) \cdot \mathcal{E}(-\widetilde{q}, \widetilde{p}_B, \widetilde{p}_{B'})$, goes to:

$$\begin{aligned} \mathcal{E}(\widetilde{q}, \widetilde{p}_A, \widetilde{p}_{A'}) \cdot \mathcal{E}(-\widetilde{q}, \widetilde{p}_B, \widetilde{p}_{B'}) &= \mathcal{E}(\widetilde{p}_A, \widetilde{p}_{A'}, \widetilde{q}) \cdot \mathcal{E}(-\widetilde{q}, \widetilde{p}_B, \widetilde{p}_{B'}) \\ &= \mathcal{E}(\widetilde{p}_A, \widetilde{p}_{A'}) \cdot \mathcal{E}(\widetilde{p}_B, \widetilde{p}_{B'}). \end{aligned} \tag{36}$$

However, in the forward scattering limit, the direction of the particles does not change, i.e. $\mathbf{p}_A \parallel \mathbf{p}'_A$. For space-space noncommutativity $p_A \wedge p'_A$ goes to zero, i.e., the phase factor goes to 1.

Thus, when the invariant momentum transfer $t = -(p'_A - p_A)^2$ goes to zero, using the properties of the polarization vectors in the Appendix B, the $S_\star$ matrix element can be shown to approach the same form as in the corresponding commutative quantity which is easily calculated in a well chosen[‖]

[¶] This is to define the charge and gravitational mass as a monopole, not as a multipole moments.

[‖] The coordinate system in which $q \cdot p_A = q \cdot p_B = 0$ [11].

coordinate system,

$$\frac{\delta_{\sigma_A\sigma_{B'}}\delta_{\sigma_B\sigma_{B'}}}{4\pi^2 E_A E_B t}\left[e_A e_B(p_A\cdot p_B)+8\pi G\, g_A g_B\left\{(p_A\cdot p_B)^2-\frac{1}{2}m_A^{\,2}m_B^{\,2}\right\}\right]. \tag{37}$$

This coincidence is quite special in the sense that the $S_\star$ matrix elements are quite different from the S matrix elements in the commutative theory when there is a momentum transfer. If particle B is at rest, this gives

$$\frac{\delta_{\sigma_A\sigma_{A'}}\delta_{\sigma_B\sigma_{B'}}}{\pi t}\left[-\frac{e_A e_B}{4\pi}+G\cdot g_A\left(2E_A-\frac{m_A^{\,2}}{E_A}\right)\cdot g_B m_B\right]. \tag{38}$$

Hence, one can interpret the coupling constant e_A as the usual charge of the particle A. Moreover one can identify the effective gravitational mass of A as

$$(m_g)_A = g_A\left(2E_A-\frac{m_A^{\,2}}{E_A}\right). \tag{39}$$

For nonrelativistic limit the g_A can be interpreted as a ratio of the gravitational mass and the inertial mass, i.e.,

$$(m_g)_A = g_A\cdot m_A, \qquad E_A\simeq m_A. \tag{40}$$

Consequently, if g_A does not depend on the species of A, it suggests that the equivalence principle holds.

3.2 Conservation law

Consider a S matrix element $S(\alpha\to\beta)$ for some reaction $\alpha\to\beta$, where the states α and β consist of various species of particles. The same reaction with emitting a soft massless particle of spin s (photon or graviton for $s=1$ or 2), momentum $\mathbf{q}$ and helicity $\pm s$ can occur. We denote the corresponding S matrix element as $S^{\pm s}(\mathbf{q},\alpha\to\beta)$. Each amplitude of this process breaks the Lorentz symmetry because a massless fields of $(s/2,s/2)$ representation break the symmetry [28]. However, a real physical reaction, to which a S-matrix element (the sums of each amplitude) correspond, should be Lorentz invariant. By requiring this condition, Weinberg could obtain the conservation relations. In this section, we investigate the similar relations by requiring the twist invariance of the $S_\star$ matrix elements.

Suppose that a soft particle is emitted by ith particle ($i=1,\cdots,n$). Then by the polology of the conventional field theory, the S matrix elements will have poles at $|\mathbf{q}|=0$ when an extra soft massless particle is emitted by one of the external lines:

$$\frac{1}{(p_i+\eta_i\cdot q)^2+m_i^{\,2}}=\frac{\eta_i}{2p_i\cdot q}, \tag{41}$$

where $\eta_i=1(-1)$ for the emission of a soft particle from an out(in) particle, respectively. By utilizing the above relation, one can obtain the S matrix elements for the soft massless particles in the $|\mathbf{q}|\to 0$

limit [11]:

$$S^{\pm s}(\mathbf{q}, \alpha \to \beta) = \frac{1}{(2\pi)^{3/2}\sqrt{2|\mathbf{q}|}} \left(\sum_i \eta_i \mathbf{e}_{\mathbf{s}i} \frac{[p_i \cdot \epsilon_{\pm}^{*}(\hat{q})]^s}{(p_i \cdot q)} \right) S(\alpha \to \beta). \tag{42}$$

In the noncommutative case, from the result of the section (2) and by using the above relation, one can deduce the $S_\star$ matrix elements for the same process in the noncommutative spacetime:

$$\begin{aligned} S_\star^{\pm s}(\mathbf{q}, \alpha \to \beta) &= \sum_i \mathcal{E}_i(\widetilde{q}, \alpha \to \beta) \cdot S_i^{\pm}(\mathbf{q}, \alpha \to \beta) \\ &= \sum_i \mathcal{E}(\widetilde{q}, \widetilde{p_i}, \widetilde{p_i} + \widetilde{q}) \cdot \mathcal{E}_I(\widetilde{p_1}, \cdots, \widetilde{p_i} + \eta_i \widetilde{q}, \cdots, \widetilde{p_n}) \cdot S_i^{\pm}(\mathbf{q}, \alpha \to \beta) \\ &= \sum_i \frac{\mathcal{E}(q, \widetilde{p_i})}{(2\pi)^{3/2}\sqrt{2|\mathbf{q}|}} \left(\eta_i \mathbf{e}_{\mathbf{s}i} \frac{[p_i \cdot \epsilon_{\pm}^{*}(\hat{q})]^s}{(p_i \cdot q)} \right) \\ &\qquad \times \mathcal{E}_I(\widetilde{p_1}, \cdots, \widetilde{p_i} + \eta_i \widetilde{q}, \cdots, \widetilde{p_n}) \cdot S_i^{\pm}(\alpha \to \beta), \end{aligned} \tag{43}$$

where $\mathcal{E}_I^i = \mathcal{E}_I(\widetilde{p_1}, \cdots, \widetilde{p_i} + \widetilde{q}, \cdots, \widetilde{p_n})$ are the phase factors in the internal process**. Since all the $\mathcal{E}_I^i$'s are the same when $q \to 0$ ($\mathcal{E}_I^i = \mathcal{E}_I$), one obtains in that limit,

$$S_\star^{\pm s}(\mathbf{q}, \alpha \to \beta) = \frac{1}{(2\pi)^{3/2}\sqrt{2|\mathbf{q}|}} \left(\sum_i \eta_i \mathbf{e}_{\mathbf{s}i} \frac{[p_i \cdot \epsilon_{\pm}^{*}(\hat{q})]^s}{(p_i \cdot q)} \right) S_\star(\alpha \to \beta), \tag{44}$$

where we used the relation $S_\star(\alpha \to \beta) = \mathcal{E}_I \cdot S(\alpha \to \beta)$. From the transformation properties of the $S_\star$ matrix elements, Eq.(28), we obtain,

$$S_\star^{\pm s}(\mathbf{q}, \alpha \to \beta) \;\; \to \;\; \frac{1}{\sqrt{2|\mathbf{q}|}} \epsilon_{\pm}^{\mu_1 *}(\hat{q}) \cdots \epsilon_{\pm}^{\mu_s *}(\hat{q}) (M_\star^{\pm})_{\mu_1 \cdots \mu_s}(\mathbf{q}, \alpha \to \beta). \tag{45}$$

Identifying the $S_\star$ matrix elements in Eq.(44) and Eq.(45) gives the invariant $M_\star$ functions for spin 1 and 2:

$$\begin{aligned} M_\star^{\mu}(\mathbf{q}, \alpha \to \beta) &= \frac{1}{(2\pi)^{3/2}} \left(\sum_i \frac{e_i \cdot \eta_i {p_i}^{\mu}}{(p_i \cdot q)} \right) S_\star(\alpha \to \beta), \\ M_\star^{\mu\nu}(\mathbf{q}, \alpha \to \beta) &= \frac{8\pi G}{(2\pi)^{3/2}} \left(\sum_i \frac{g_i \cdot \eta_i {p_i}^{\mu} {p_i}^{\nu}}{(p_i \cdot q)} \right) S_\star(\alpha \to \beta). \end{aligned} \tag{46}$$

Requiring the twist invariance to the $S_\star$ matrix elements, Eq.(33), gives:

$$\begin{aligned} 0 = q_\mu M_\star^{\mu}(\mathbf{q}, \alpha \to \beta) &\;\; \to \;\; \sum_i \eta_i e_i = 0, \\ 0 = q_\mu M_\star^{\mu\nu}(\mathbf{q}, \alpha \to \beta) &\;\; \to \;\; \sum_i g_i \cdot \eta_i p_i^{\nu} = 0, \end{aligned} \tag{47}$$

** The integrals over the internal loop momenta have been suppressed because they do not affect the final result since the change in the phase factor occurs only in the external lines.

in general. Hence, one obtains the charge conservation law for the spin 1 fields. For $s = 2$, in order to satisfy the two relations, $\sum \eta_i p_i = 0$ (4-momentum conservation) and $\sum_i g_i \cdot \eta_i p_i^\nu = 0$, g_i should be constants (i.e. independence of the particle species). The universality of this coupling constant as a ratio of gravitational mass and inertial mass shows that the equivalence principle is satisfied even in the noncommutative spacetime.

4 Discussion

We have found that the conservation of charge and the equivalence principle are satisfied even in the canonical noncommutative spacetime. The derivation was fairly straightforward, once we have constructed the $S_\star$ matrix for general field theories. The assumption was that the quanta of the gravitation is massless spin 2, massless spin 1 for photon.

We extended the construction of the $S_\star$ matrix to general fields, especially for the massless fields of integer spin. The twisted Feynman diagrams can be constructed by the same irreducible representations as those in the untwisted theories with the same rule except for the different phase factors at each vertex. Hence, we can say that the same reasoning apply to the massive fields.

We use the $(s/2, s/2)$ representation for the field operators mainly because one can obtain the condition, Eq.(33), requiring the twist Lorentz invariance. In this representation, since the polarization vectors are not Lorentz four vectors, each amplitude of emitting a real soft massless particle violate the symmetry. We have used this property to show the conservation law in this paper. Another representation for the fields, the $(s, 0) \oplus (0, s)$ representation, which has the parity symmetry, can be made Lorentz covariant. One realizes that due to these properties one cannot derive the conservation law by the method used in this paper.

Charge conservation in the noncommutative spacetime is expected from the gauge symmetry and the noncommutative Noether theorem. However, it is not quite certain whether the equivalence principle is satisfied in the noncommutative spacetime. Since the noncommutative spacetime is not locally Minkowski nor the local symmetry group is $SO(1, 3)$, one can not guarantee if the principle is satisfied. But in our twisted symmetry context, one can expect that the equivalence principle is satisfied because the algebra structure is the same as the conventional theory though the coalgebra structure is different. The applicability of the S matrix theoretic proof of the equivalence principle given in this paper is restrictive because the analysis is perturbative in nature. We expect that the conclusion of this paper is to be one of the stepping stone towards the further understanding of the nature of the principle in quantum gravity.

This paper shows an example for the usefulness of the twisted symmetry to derive the physically important relations in the noncommutative spacetime. We think that the twist analysis is adequate for the schematic approaches to the noncommutative physics, while the other approaches are suitable for the explicit calculations though they are equivalent. The future applications of the twisted symmetry to the other issues in this direction of approach are expected.

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A Exact calculation of the polarization tensor for massless fields

The creation and annihilation operators of the massless field transform under the Lorentz transformation as [28]

$$\begin{aligned} a^\dagger(\Lambda p, \sigma) &= e^{-i\sigma\Theta[W(\Lambda,p)]}\, U(\Lambda)\, a^\dagger(p,\sigma)\, U^{-1}(\Lambda), \\ a(\Lambda p, \sigma) &= e^{+i\sigma\Theta[W(\Lambda,p)]}\, U(\Lambda)\, a(p,\sigma)\, U^{-1}(\Lambda), \end{aligned} \tag{48}$$

up to phase factors, where $W(\Lambda, p) = \mathcal{L}^{-1}(p)\Lambda^{-1}\mathcal{L}(\Lambda p)$ denotes the Wigner rotation defined as in [28] (Appendix C), and the $\Theta[W(\Lambda, p)]$ is the related angle to which the little group corresponds. Hereafter, we abbreviate $\Theta[W(\Lambda, p)]$ as $\Theta(\Lambda, p)$. It is to be noted that we use $U(\Lambda)$ instead of $U_\theta(\Lambda)$ in here. This follows immediately from the properties of the twist in section 2.1.

The field $\hat{\Psi}_\theta{}^A$ transform as

$$[D^s(\Lambda^{-1})]^A_{\ B}\, \hat{\Psi}_\theta{}^B(\Lambda x) = U(\Lambda)\, \hat{\Psi}_\theta{}^A(x)\, U^{-1}(\Lambda). \tag{49}$$

In order to satisfy the two relations (48), (49), the polarization tensor should transform as

$$\epsilon_\theta{}^A(p,\sigma) = e^{-i\sigma\Theta(\Lambda,p)}\, [D^s(\Lambda^{-1})]^A_{\ B}\, \epsilon_\theta{}^B(\Lambda p,\sigma). \tag{50}$$

When $\Lambda = W$ and $p = k$, the above relation goes to

$$[D^s(W)]^A_{\ B}\, \epsilon_\theta{}^B(k,\sigma) = e^{-i\sigma\Theta(W,p)}\epsilon_\theta{}^A(k,\sigma), \tag{51}$$

where k denotes the standard momentum. The Wigner rotation can be written as $W(\phi, \alpha, \beta) = R(\phi)T(\alpha, \beta)$ because the little group is isomorphic to $ISO(2)$ for massless fields [28].

Suppose that the field transforms as (m, n) representation of spin s $(m + n = s)$. When the ϕ, α and β are infinitesimal, $D^s(W)$ can be written as

$$D^s(W) \simeq 1 - i\phi(M_3 + N_3) + (\alpha + i\beta)(M_1 - iM_2) + (\alpha - i\beta)(N_1 + iN_2). \tag{52}$$

For $M_- \equiv M_1 - iM_2$, $N_+ \equiv M_1 + iM_2$, and $\Theta \to \phi$ we have:

$$\begin{aligned} (M_3 + N_3)\, \epsilon_\theta(k,\sigma) &= \sigma\, \epsilon_\theta(k,\sigma), \\ M_-\, \epsilon_\theta(k,\sigma) &= 0, \\ N_+\, \epsilon_\theta(k,\sigma) &= 0, \end{aligned} \tag{53}$$

which gives

$$\begin{aligned} M_3\,\epsilon_\theta(k,\sigma) &= -m\,\epsilon_\theta(k,\sigma), \\ N_3\,\epsilon_\theta(k,\sigma) &= +n\,\epsilon_\theta(k,\sigma). \end{aligned} \tag{54}$$

The $\epsilon_\theta(k,\sigma)$ satisfys the same equation as $\epsilon(k,\sigma)$. Since the highest or lowest weight corresponds to a unique state, one obtains the same polarization tensor as a solution, $\epsilon_\theta(k,\sigma)=\epsilon(k,\sigma)$.

B Properties of the polarization vectors

Here, we summarize the properties of the polarization vector. The properties of the polarization tensors of other rank, Eq.(24), follows from it.

Solving the (53) for $\sigma=\pm 1$ gives the explicit form of the polarization vector for the standard momentum as

$$\epsilon_\pm{}^\mu(k) \equiv \frac{1}{\sqrt{2}}(1,\pm i,0,0), \tag{55}$$

where we made the conventional choice of the phase. The polarization vector of momentum p is defined as

$$\epsilon_\pm{}^\mu(p) = [\mathcal{L}(p)]^\mu{}_\nu\,\epsilon_\pm{}^\nu(k), \tag{56}$$

where $\mathcal{L}(p)$ is the Lorentz transformation which takes k to p, i.e., $p^\mu=[\mathcal{L}(p)]^\mu{}_\nu\,k^\nu$. Then the well known properties of the polarization vectors can be deduced [11]:

$$\begin{aligned} p_\mu\epsilon_\pm{}^\mu(\hat{p}) &= 0, & (57)\\ \epsilon^*_{\pm\mu}(\hat{p})\epsilon_\pm{}^\mu(\hat{p}) &= 1, \qquad \epsilon_{\pm\mu}(\hat{p})\epsilon_\pm{}^\mu(\hat{p})=0, & (58)\\ \epsilon_\pm{}^{\mu *}(\hat{p}) &= \epsilon_\mp{}^\mu(\hat{p}), \qquad \epsilon_\pm{}^0(\hat{p})=0, & (59)\\ \sum_\pm \epsilon_\pm{}^\mu(\hat{p})\epsilon_\pm{}^{\nu *}(\hat{p}) &= \eta^{\mu\nu}+(\widetilde{p}^\mu p^\nu+\widetilde{p}^\nu p^\mu)/2|\mathbf{p}|^2 \equiv \Pi^{\mu\nu},\ \widetilde{p}\equiv(|\mathbf{p}|,-\mathbf{p}), & (60)\\ \sum_\pm \epsilon_\pm{}^{\mu_1}(\hat{p})\epsilon_\pm{}^{\mu_2}(\hat{p})\epsilon_\pm{}^{\nu_1 *}(\hat{p})\epsilon_\pm{}^{\nu_2 *}(\hat{p}) &= \frac{1}{2}\left\{\Pi^{\mu_1\nu_1}(\hat{p})\Pi^{\mu_2\nu_2}(\hat{p})+\Pi^{\mu_1\nu_2}(\hat{p})\Pi^{\mu_2\nu_1}(\hat{p})\right. \\ &\qquad \left.-\Pi^{\mu_1\mu_2}(\hat{p})\Pi^{\nu_1\nu_2}(\hat{p})\right\}. & (61) \end{aligned}$$

The polarization 'vectors' are not the Lorentz four vectors[††], rather they transform as

$$\begin{aligned} \epsilon^\mu_\pm(p) &= e^{\mp i\Theta(\Lambda,p)}[D^s_\epsilon(\Lambda,p)]^\mu{}_\nu\epsilon^\nu_\pm(\Lambda p), \\ [D^s_\epsilon(\Lambda,p)]^\mu{}_\nu &= (\Lambda^{-1})^\mu{}_\nu-\frac{p^\mu}{|\mathbf{p}|}(\Lambda^{-1})^0{}_\nu. \end{aligned} \tag{62}$$

For general spin s, the polarization tensor can be written as

$$\epsilon^A_\sigma(p) = e^{\mp i\Theta(\Lambda,p)}[D^s_\epsilon(\Lambda,p)]^A{}_B\,\epsilon^B_\sigma(\Lambda p), \tag{63}$$

up to a phase factor.

[††]It comes from the fact that the little group is not semisimple for massless fields. Translation in the little group isomorphic to $ISO(2)$ generates the gradient term in the transformations.

C Invariant $M_\star$ function for massless field

Let P denotes a shorthand notation for the external lines, $P \equiv (p_1, \ldots, p_n)$, and K denotes a standard momenta[‡‡] $K \equiv (k_1, \ldots, k_n)$. There exists a unique Lorentz transformation satisfying $P \equiv L_P K$. Then the relation between the Lorentz group and the little group can be described symbolically as:

$$
\begin{tikzcd}
K \arrow[d, "L_P"'] & K \arrow[l, "W(\Lambda{,}P)"'] \arrow[d, "L_{\Lambda P}"] \\
P \arrow[r, "\Lambda"'] & \Lambda P
\end{tikzcd}
\quad ,
$$

where $W(\Lambda, P)$ is the Wigner transformation to which a Lorentz transformation Λ and the momenta P correspond. Since the twist do not change the group properties, above relation also holds for the twisted symmetry group. The $S_\star$ matrix elements transform as

$$
\begin{aligned}
S_\star[P] &= D^s_\theta[W(\Lambda, P)]\; S_\star[\Lambda P] \\
&= D^s_\theta[L_P^{-1} \cdot \Lambda^{-1} \cdot L_{\Lambda P}]\; S_\star[\Lambda P]
\end{aligned} \tag{64}
$$

where the indices for the external lines are suppressed. From the golden rule and (48), the explicit transformation formular for $S_\star[P]$ can be obtained:

$$
S_\star[P] = \frac{N(\Lambda P)}{N(P)}\; e^{\pm is\Theta(\Lambda,P)}\; S_\star[\Lambda P], \tag{65}
$$

where N denotes the corresponding normalization factor.

If one defines $M_\star[P]$ as $S_\star[P] = D^s_\theta[L_P^{-1}]\; M_\star[P]$, one can show that $M_\star[P]$ transform as

$$
M_\star[P] = D^s_\theta[\Lambda^{-1}]\; M_\star[\Lambda P], \tag{66}
$$

i.e., it is twist invariant. We will call it as an invariant $M_\star$ function as in [29, 30].

Let us find out the invariant $M_\star$ function for massless fields. First, we define the quanties for the standard momenta K. The choice for $M_\star[K]$ is

$$
M^A_\star[K] = N(K)\; \epsilon^A[K]\; S_\star[K]. \tag{67}
$$

If we define $M_\star[P]$ as

$$
M^A_\star[P] = [D^s_\theta(L_P)]^A{}_B\; M^B_\star[K], \tag{68}
$$

then one can easily show that $M^A_\star[P]$ is twist invariant.

By using the explicit form of D^s_ϵ in (62) and $k_{b_j} M^{b_1\cdots b_s}_\star[K] = 0$, one obtains the relation

$$
\begin{aligned}
\epsilon^C[P]^*\; (M_\star)_C[P] &= \left(e^{\pm is\Theta(L_P^{-1},P)}[D^\epsilon_\theta(L_P^{-1})]^C{}_A\; \epsilon^A[K]^*\right)\left([D_\theta(L_P)]_C{}^B\; (M_\star)_B[K]\right) \\
&= e^{\pm is\Theta(L_P^{-1},P)}\; \epsilon^A[K]^*\; [D_\theta(L_P^{-1}) \cdot D^\epsilon_\theta(L_P^{-1})]^B{}_A\; (M_\star)_B[K] \\
&= e^{\pm is\Theta(L_P^{-1},P)}\; \epsilon^A[P]^*\; (M_\star)_A[P] \\
&= e^{\pm is\Theta(L_P^{-1},P)} N(K)\; S_\star[K],
\end{aligned} \tag{69}
$$

[‡‡] see [11],[29].

where we used the relation

$$\begin{aligned}
[D_\theta(L_P^{-1})\cdot D_\theta^\epsilon(L_P^{-1})]^B{}_A\,(M_\star)_B[K] &\equiv [D(L_P^{-1})\cdot D_\epsilon(L_P^{-1})]^B{}_A\,(M_\star)_B[K] \\
&= \left(\delta^{b_1}{}_{a_1}-\frac{k^{b_1}}{|\mathbf{p}|}(L_P)^0{}_{a_1}\right)\cdots\left(\delta^{b_s}{}_{a_s}-\frac{k^{b_s}}{|\mathbf{p}|}(L_P)^0{}_{a_s}\right)\times(M_\star)_{b_1\cdots b_s}[K] \\
&= (M_\star)_{a_1\cdots a_s}[K]. \qquad (70)
\end{aligned}$$

By setting $\Lambda = L_P^{-1}$ in (65), one can see that (69) is $S_\star[P]$. Thus, the desired form of $S_\star[P]$ is

$$S_\star[P] = N(P)\,\epsilon_A^*[P]\,M_\star^A[P]. \qquad (71)$$

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